21/4/21

Φυλλάδιο Ασκήσεων #7

1) $A=\begin{pmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{pmatrix}$: Πραγματικός συμμετρικός $\Rightarrow$ Διαγωνοποιείται.

$$\chi_A(\lambda)=\det\begin{pmatrix}1-\lambda & -1 & 0\\ -1 & 2-\lambda & -1\\ 0 & -1 & 1-\lambda\end{pmatrix}\xrightarrow[\Sigma_1\to\Sigma_1+\Sigma_3]{\Sigma_1\to\Sigma_1+\Sigma_2}\det\begin{pmatrix}-\lambda & -1 & 0\\ -\lambda & 2-\lambda & -1\\ -\lambda & -1 & 1-\lambda\end{pmatrix}=$$

$$=-\lambda\det\begin{pmatrix}1 & -1 & 0\\ 1 & 2-\lambda & -1\\ 1 & -1 & 1-\lambda\end{pmatrix}\xrightarrow{\Sigma_2\to\Sigma_2+\Sigma_1}-\lambda\det\begin{pmatrix}1 & 0 & 0\\ 1 & 3-\lambda & -1\\ 1 & 0 & 1-\lambda\end{pmatrix}=$$

$$=-\lambda\cdot 1\det\begin{pmatrix}3-\lambda & -1\\ 0 & 1-\lambda\end{pmatrix}=-\lambda(3-\lambda)\cdot(1-\lambda)=0\Rightarrow \begin{aligned}\lambda_1&=0\\ \lambda_2&=1\\ \lambda_3&=3\end{aligned}$$

$$V(0):\begin{pmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\Rightarrow \begin{aligned}x-y&=0\\ -y+z&=0\end{aligned}\Rightarrow x=y=z$$

$V(0)$: $(x,x,x)=x(1,1,1)$ $\quad V(0)=\langle(1,1,1)\rangle=\langle\frac{1}{\sqrt3}(1,1,1)\rangle$

$V(1)=\langle\frac{1}{\sqrt2}(-1,0,1)\rangle$, $V(3)=\langle\frac{1}{\sqrt6}(1,-2,1)\rangle$

$$P=\begin{pmatrix}1/\sqrt3 & -1/\sqrt2 & 1/\sqrt6\\ 1/\sqrt3 & 0 & -2/\sqrt6\\ 1/\sqrt3 & 1/\sqrt2 & 1/\sqrt6\end{pmatrix}\text{ ορθογώνιος}\quad \Lambda=\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 3\end{pmatrix}$$

$A=P\Lambda P^t$

2) Θεωρ. αναπαρ. Riesz $T: \mathbb{R}_2[x] \to \mathbb{R}$ με τύπο $T(f) = f(0)$
$f(x) = ax^2 + bx + \gamma \Rightarrow T(f) = \gamma$. Ν.δ.ο $T(f+f')$ γραμμική

$T(f+f') = T((a+a')x^2 + (b+b')x + \gamma+\gamma') = \gamma+\gamma' = T(f) + T(f')$
$T(kf) = T(kax^2 + kbx + k\gamma) = k\gamma = k \cdot T(f)$
γραμμική

Ν.β. ο $\ker T$: $\ker T$: $T(f) = 0 \Rightarrow \gamma = 0 \Rightarrow f(x) = ax^2 + bx \Rightarrow$
$\Rightarrow \ker T = \langle x^2, x \rangle$ $\qquad$ T είναι επί
Βρείτε $g(x) = ax^2 + bx + \gamma$ ώστε $g \perp \ker T \Leftrightarrow g \perp x^2, g \perp x$
$\langle g, x^2 \rangle = 0 = \langle g, x \rangle \Leftrightarrow \int_0^1 g(x)\, x^2 dx = 0 = \int_0^1 g(x) \cdot x\, dx$

$\int_0^1 (ax^4 + bx^3 + \gamma x^2) dx = 0 \Rightarrow \frac{a}{5} + \frac{b}{4} + \frac{\gamma}{3} = 0 \Rightarrow 12a + 15b + 20\gamma = 0$ (1)

$\int_0^1 (ax^3 + bx^2 + \gamma x) dx = 0 \Rightarrow \frac{a}{4} + \frac{b}{3} + \frac{\gamma}{2} = 0 \Rightarrow 3a + 4b + 6\gamma = 0$
$\xRightarrow{\cdot(-4)} -12a - 16b - 24\gamma = 0$ (2)

(2) $\xRightarrow{+(1)}$ $-b - 4\gamma = 0 \Rightarrow b = -4\gamma \xRightarrow{(2)} 3a - 16\gamma + 6\gamma = 0 \Rightarrow a = \frac{10}{3}\gamma$

$g(x) = \frac{10}{3}\gamma x^2 - 4\gamma x + \gamma = \gamma\left(\frac{10}{3}x^2 - 4x + 1\right)$

Για $\gamma = 1 \Rightarrow \boxed{g(x) = \frac{10}{3}x^2 - 4x + 1}$

Ν.β. το $h(x) = \frac{T(g)}{\langle g, g \rangle} \cdot g$

$T(g) = g(0) = 1$
$\|g\|^2 = \int_0^1 g^2 dx = \int_0^1 \left(\frac{10}{3}x^2 - 4x + 1\right)^2 dx = \int_0^1 \left[\frac{100}{9}x^4 + 16x^2 + 1 - \frac{80}{3}x^3 + \right.$
$\left. + \frac{20}{3}x^2 - 8x\right] dx = \frac{20}{9} + \frac{16}{3} + 1 - \frac{20}{3} + \frac{20}{9} - 4 =$
$= \frac{1}{9}(20 + 48 + 9 - 60 + 20 - 36) = \frac{1}{9}$

$$h(x) = \frac{1}{\frac{1}{9}} \cdot g(x) = 9\left(\frac{10}{3}x^2 - 4x + 1\right)$$

Να δείξετε ότι $T(f(x)) = \langle f(x), h(x) \rangle$

Αν $f(x) = \alpha x^2 + \beta x + \gamma \Rightarrow T(f) = \gamma$

$$\int_0^1 (\alpha x^2 + \beta x + \gamma)\, 9\left(\frac{10}{3}x^2 - 4x + 1\right) dx = \cancel{9\int_0^1 \frac{10}{3}\alpha x^4 - 4\alpha x^3 + \alpha x^2}$$

$$= 9\int_0^1 \left(\frac{10}{3}\alpha x^4 - 4\alpha x^3 + \alpha x^2 + \frac{10}{3}\beta x^3 - 4\beta x^2 + \beta x + \frac{10}{3}\gamma x^2 - 4\gamma x + \gamma\right) dx$$

$$= 9\left(\frac{2\alpha}{3} - \alpha + \frac{\alpha}{3} + \frac{5}{6}\beta - \frac{4}{3}\beta + \frac{\beta}{2} + \frac{10\gamma}{9} - 2\gamma + \gamma\right) =$$

$$= 9\left(\cancel{\frac{2\alpha - 3\alpha + \alpha}{3}}^{0} + \cancel{\frac{5\beta - 8\beta + 3\beta}{6}}^{0} + \frac{10\gamma - 9\gamma}{9}\right) = 9 \cdot \frac{1}{9}\gamma = \gamma$$

3) $A = \begin{pmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{pmatrix}$: πραγματικός συμμετρικός $\Rightarrow$ Διαγωνοποιείται

$$\chi_A(\lambda) = \det\begin{pmatrix} -2-\lambda & 0 & -36 \\ 0 & -3-\lambda & 0 \\ -36 & 0 & -23-\lambda \end{pmatrix} = (-3-\lambda)\det\begin{pmatrix} -2-\lambda & -36 \\ -36 & -23-\lambda \end{pmatrix} =$$

$$= -(3+\lambda)\left[46 + 2\lambda + 23\lambda + \lambda^2 - 36^2\right] = -(3+\lambda)(\lambda^2 + 25\lambda + 46 - 36^2)$$

$$= -(3+\lambda)(\lambda - 25)\cdot(\lambda + 50)$$

$\lambda_1 = -3$

$\lambda_2 = 25$

$\lambda_3 = -50$

$V(-3) = \langle (0,1,0) \rangle$, $V(25) = \langle (-4,0,3) \rangle$, $V(-50) = \langle (3,0,4) \rangle$

↑ κανονικό

$= \langle \frac{1}{5}(-4,0,3) \rangle$ $\quad = \langle \frac{1}{5}(3,0,4) \rangle$

$$P = \begin{pmatrix} 0 & -4/5 & 3/5 \\ 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \end{pmatrix} \quad \Lambda = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & -50 \end{pmatrix} \quad A = P\Lambda P^t$$

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5) $A=\begin{pmatrix} 0 & i & 0 \\ -i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ εξετάστε αν διαγωνοποιείται. και αν ναι, πως?

Ο A είναι ερμιτιανός ($A=\bar{A}^t$) $\Rightarrow$ διαγωνοποιείται

$\chi_A(\lambda)=\begin{pmatrix} -\lambda & i & 0 \\ -i & -\lambda & -i \\ 0 & i & -\lambda \end{pmatrix} = -\det\begin{pmatrix} +\lambda & i & 0 \\ i & \lambda & i \\ 0 & i & \lambda \end{pmatrix} = -\det\begin{pmatrix} -\lambda & i & 0 \\ 0 & \lambda & i \\ \lambda & i & -\lambda \end{pmatrix} =$

$= -\lambda\det\begin{pmatrix} -1 & i & 0 \\ 0 & \lambda & i \\ 1 & i & -\lambda \end{pmatrix} = -\lambda\det\begin{pmatrix} -1 & i & 0 \\ 0 & \lambda & i \\ 0 & 2i & -\lambda \end{pmatrix} = \lambda\det\begin{pmatrix} \lambda & i \\ 2i & -\lambda \end{pmatrix} =$

$= \lambda(-\lambda^2-2i^2) = \lambda(-\lambda^2+2) = -\lambda(\lambda^2-2) \Rightarrow \lambda_1=0$

$\lambda_2=-\sqrt{2}$

$\lambda_3=\sqrt{2}$

$V(0)=\langle(-1,0,1)\rangle$, $V(-\sqrt{2})=\langle(1,i\sqrt{2},1)\rangle$, $V(\sqrt{2})=\langle(1,-i\sqrt{2},1)\rangle$

$=\langle\frac{1}{\sqrt{2}}(-1,0,1)\rangle$ $\quad=\langle\frac{1}{2}(1,i\sqrt{2},1)\rangle$ $\quad=\langle\frac{1}{2}(1,-i\sqrt{2},1)\rangle$

$P=\begin{pmatrix} -1/\sqrt{2} & 1/2 & 1/2 \\ 0 & i\sqrt{2}/2 & -i\sqrt{2}/2 \\ 1/\sqrt{2} & 1/2 & 1/2 \end{pmatrix}$ $\quad \Lambda=\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$

Παρατήρηση: Αν ο $A=\begin{pmatrix} 0 & i & 0 \\ -i & 0 & i \\ 0 & i & 0 \end{pmatrix}$ $\leftarrow$ δεν είναι ερμιτιανός. έχει 1 ιδιοτιμή $\lambda=0$ $V(0)=\langle(1,0,0)\rangle$ δεν διαγωνοποιείται

$\|(1,i\sqrt{2},1)\|^2=\langle(1,i\sqrt{2},1),(1,i\sqrt{2},1)\rangle = 1\cdot\bar{1}+i\sqrt{2}(\overline{i\sqrt{2}})+1\cdot 1\ldots$

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4) $T: \mathbb{R}^3 \Rightarrow \mathbb{R}^3 \quad T(x,y,z) = (ax-y+z, -x+by-z, x-y+2z)$

Ν.δ. $a,b \in \mathbb{R}$: $\dim V(1) = 2$

k = κανονική βάση

$$T_k^k = \begin{pmatrix} a & -1 & 1 \\ -1 & b & -1 \\ 1 & -1 & 2 \end{pmatrix}$$ Πραγμ. συμ. ⇒ Διαγωνοποιείται ⇒ $\dim V(1) =$ = πολλ/τα $1 = 2$

$$V(1): \begin{pmatrix} a-1 & -1 & 1 \\ -1 & b-1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \dim\ker \underbrace{\begin{pmatrix} a-1 & -1 & 1 \\ -1 & b-1 & 1 \\ 1 & -1 & 1 \end{pmatrix}}_{B} = 2$$

$$\Rightarrow \operatorname{rank} \underbrace{\begin{pmatrix} a-1 & -1 & 1 \\ -1 & b-1 & 1 \\ 1 & -1 & 1 \end{pmatrix}}_{B}$$

$A: \mathbb{R}^n \Rightarrow \mathbb{R}^n$, $\dim\ker A + \dim \operatorname{Im} A = n$

$\dim \operatorname{Im} A = \operatorname{rank} A$

$\dim\ker(B) + \operatorname{rank}(B) = 3 \Rightarrow \operatorname{rank}(B) = 1$

$$1 \le \operatorname{rank} \begin{pmatrix} a-1 & -1 & 1 \\ -1 & b-1 & 1 \\ 1 & -1 & 1 \end{pmatrix} = 1 \le 3$$

$(a-1, -1, 1)$ και $(-1, b-1, 1)$ είναι γρ. εξ. από την $(1,-1,1) \Leftrightarrow (a-1,-1,1) = t(1,-1,1)$ και $(-1,b-1,1) = s(1,-1,1)$

$t=1 \Rightarrow a-1=1 \Rightarrow a=2$ $\quad\quad \Rightarrow s=1 \Rightarrow b-1=1 \Rightarrow$
$\Rightarrow b=2$

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \quad \chi_A(\lambda) = \det \begin{pmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{pmatrix} =$$

$$= \det \begin{pmatrix} 2-\lambda & 1 & 1 \\ -1 & \lambda-2 & -1 \\ 1 & 1 & 2-\lambda \end{pmatrix} = \det \begin{pmatrix} 4-\lambda & 1 & 1 \\ \lambda-4 & \lambda-2 & -1 \\ 4-\lambda & 1 & 2-\lambda \end{pmatrix} =$$

$$= (4-\lambda)\det\begin{pmatrix} 1 & 1 & 1 \\ -1 & \lambda-2 & -1 \\ 1 & 1 & 2-\lambda \end{pmatrix} \Rightarrow \lambda_1=\lambda_2=1, \quad \lambda_3 = 4$$

$$V(1) \Rightarrow B\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow x - y + z = 0 \Rightarrow y = x + z$$

$$V(1) = \langle (x, x+z, z) \rangle = \langle (1,1,0), (0,1,1) \rangle :$$

$$= (0,1,1) - \frac{\langle (0,1,1), (1,1,0) \rangle}{\langle (1,1,0), (1,1,0) \rangle}(1,1,0) = (0,1,1) - \frac{1}{2}(1,1,0) =$$

$$= \left(-\frac{1}{2}, \frac{1}{2}, 1\right)$$

$$V(1) = \left\langle \frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{6}}(-1,1,2) \right\rangle$$

$$V(4) : \begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow -2x - y + z = 0 \Rightarrow z = 2x + y$$

$$x - 2y - z = 0 \Rightarrow$$

$$\Rightarrow x + 2y + 2x + y = 0$$

$$x = -y, \quad z = -2y + y =$$

$$= -y$$

$$V(4) = \langle (-y, y, -y) \rangle = \left\langle \frac{1}{\sqrt{3}}(-1,1,-1) \right\rangle$$

$$T_k^k = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & -2 \end{pmatrix} = P \Lambda P^t = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix} = \ldots$$

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Ν.δ. πίνακας Γ ώστε $\Gamma^m = A$, $m \geq 1$

$A = P \cdot \Lambda P^t = \Gamma^m$

Για $\Gamma = P \begin{pmatrix} \sqrt[m]{1} & 0 & 0 \\ 0 & \sqrt[m]{1} & 0 \\ 0 & 0 & \sqrt[m]{4} \end{pmatrix} P^t = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt[m]{4} \end{pmatrix} P^t \Rightarrow \Gamma^m = A$

6) Έχω ότι $A = A^t$, $B = B^t$ και $A, B > 0$. Ν.δ.ο $aA + bB = (aA+bB)^t$ (1)

$(aA+bB)^t$

και $aA + bB > 0$ για $|a| + |b| > 0$ (2)

$A > 0 \overset{\text{ορσ}}{\Leftrightarrow} A = A^t$ και $\langle u, uA \rangle > 0 \quad \forall u \neq \bar{0}$

$(aA + bB)^t = (aA)^t + (bB)^t = aA^t + bB^t = aA + bB$ (1)

$\langle u, u(aA + bB) \rangle = \langle u, auA + buB \rangle =$

$= \langle u, auA \rangle + \langle u, buB \rangle = a\langle u, uA \rangle + b\langle u, uB \rangle > 0$ (2)

7) Να υπολογίσετε το $a \in \mathbb{R}$ ώστε $A = A^t$, $A > 0$

$A = \begin{pmatrix} 1 & a & 0 \\ a & 1 & a \\ 0 & a & 1 \end{pmatrix}$

Κριτήριο Silvester: $\Rightarrow 1 > 0$ (✓)

$\det \begin{vmatrix} 1 & a \\ a & 1 \end{vmatrix} > 0 \Rightarrow 1 - a^2 > 0 \Rightarrow \boxed{-1 < a < 1}$

$\det A > 0 \Rightarrow \det \begin{pmatrix} 1 & a & a \\ a & 1 & a \\ 0 & a & 1 \end{pmatrix} > 0 \Rightarrow 1 \det \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} - a \det \begin{pmatrix} a & 0 \\ a & 1 \end{pmatrix} > 0$

$\Rightarrow 1 - a^2 - a^2 > 0 \Rightarrow 1 - 2a^2 > 0 \Rightarrow \boxed{-\frac{1}{\sqrt{2}} < a < \frac{1}{\sqrt{2}}}$

Πού αναλύθηκαν? $-\frac{1}{\sqrt{2}} < a < \frac{1}{\sqrt{2}}$ ← Μόνο τότε ο $A > 0$

Προσαρτημένοι πίνακες – Συζυγείς

V^n είναι Ευκλείδειος χώρος $T: V^n \to V^n$ ενδομορφισμός. Ζητάμε μια γραμμική απεικόνιση $T^*: V^n \to V^n$ ώστε να ισχύει $\boxed{\langle T(u), w\rangle = \langle u, T^*(w)\rangle}$ $\forall u, w \in V^n$. Η T^* καλείται προσαρτημένη ή συζυγής της T

Π.χ.: Δίνεται η $T: \mathbb{R}^3 \to \mathbb{R}^3$ με το κανονικό εσωτερικό γινόμενο με τύπο $T(x,y,z) = (0,x,y)$. Ν.β. η προσαρτημένη απεικόνιση.
Θέλουμε $T^*(a,b,\gamma) = ?$
Πρέπει να ισχύει: $\langle T(x,y,z), (a,b,\gamma)\rangle = \langle (x,y,z), T^*(a,b,\gamma)\rangle \Rightarrow$
$\Rightarrow \langle (0,x,y), (a,b,\gamma)\rangle = \langle (x,y,z), T^*(a,b,\gamma)\rangle \Rightarrow$
$\Rightarrow \boxed{xb + y\gamma = \langle (x,y,z), T^*(a,b,\gamma)\rangle}$ $(*)$ $T^*(a,b,\gamma) = ?$

$T^*(1,0,0) = ?$, $T^*(0,1,0) = ?$, $T^*(0,0,1) = ?$
$T^*(1,0,0) = (\kappa, \lambda, \mu) \overset{(*)}{\Rightarrow} 0 = \langle (x,y,z), (\kappa,\lambda,\mu)\rangle \Rightarrow$
$\Rightarrow 0 = x\kappa + \lambda y + \mu z$ (1)
$T^*(0,1,0) = (\kappa', \lambda', \mu') \overset{(*)}{\Rightarrow} x = \langle (x,y,z), (\kappa',\lambda',\mu')\rangle \Rightarrow$
$\Rightarrow x = x\kappa' + y\lambda' + z\mu'$ (2) $\forall x,y,z \in \mathbb{R}$. Άρα $\boxed{\kappa' = 1}$ $\boxed{\lambda' = 0}$ $\boxed{\mu' = 0}$

Η (1) ισχύει για όλα τα x,y,z: $0 = x\kappa + \lambda y + \mu z$ $\forall x,y,z \in \mathbb{R}$
Για $y = z = 0 \Rightarrow 0 = x \cdot \kappa$. Αν $x \neq 0 \Rightarrow \kappa = 0$. Για τον ίδιο λόγο $\lambda = 0$ και $\mu = 0$. Άρα $T^*(1,0,0) = (0,0,0)$

(2) $\Rightarrow x = x\kappa' + y\lambda' + z\mu'$ Για $y = z = 0 \Rightarrow \kappa' = 1 \Rightarrow x = x + y\lambda' + z\mu' \Rightarrow$
$\Rightarrow 0 = y\lambda' + z\mu'$
Για $z = 0$ και $y \neq 0$ έχουμε: $0 = y\lambda' \Rightarrow \lambda' = 0$. Για τον ίδιο λόγο $\mu' = 0$
$T^*(0,1,0) = (1,0,0)$ (3). Ακριβώς με τον ίδιο τρόπο δείχνουμε ότι $T^*(0,0,1) = (0,1,0)$ (4). Άρα $T^*(a,b,\gamma) = aT^*(1,0,0) + bT^*(0,1,0) + \gamma T^*(0,0,1) \overset{(3),(4)}{=} (b,\gamma,0)$

$T(x,y,z) = (0,x,y)$

$$T_k^k = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (T^*)_k^k = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Τι σχέση έχουν αυτοί οι 2 πίνακες ? $T_k^k = \left[(T^*)_k^k\right]^t$